

## Final Exam Solutions

Problem 1

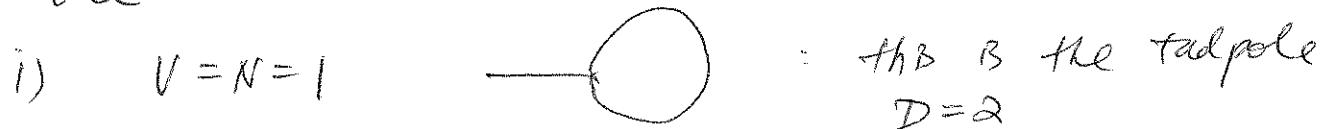
$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{3!} g \phi^3$$

(a) The potential  $V(\phi) = \frac{1}{3!} g \phi^3$  is unbounded from below, and is thus unstable.

(b) From Eq.(10.13) in P&S, in  $d=4$ ,

$$\begin{aligned} D &= 4 + [3 \cdot \frac{4-\alpha}{2} - 4] \cdot V - (\frac{4-\alpha}{2}) N \\ &= 4 - V - N. \end{aligned}$$

The only non-trivial superficially divergent diagrams are



No other diagrams are superficially divergent.

(P.2)

(C) To compute the 1-loop diagrams in  $d$ -dim, we must determine the dimensionality of the coupling constant  $g$ .

$$[\phi] = \frac{d-2}{2} \Rightarrow [g] = d - 3 \cdot \frac{d-2}{2} = 1 + (\alpha - \frac{d}{2}) = 3 - \frac{d}{2}$$

$\therefore g$  will pick up an ~~dim~~ extra  $(\frac{d}{2}-\frac{d}{2})$  dim for  $d \rightarrow d+2$   
i.e.  $g \rightarrow \frac{\mu^{-\epsilon}}{\mu^2} g$ .

Then

$$\begin{aligned} -\text{O} &= \frac{i g \mu^{\frac{d-2}{2}}}{2(2\pi)^d} \int d^d k \frac{1}{(k^2 - m^2 + i\epsilon)} \frac{1}{((p+k)^2 - m^2 + i\epsilon)} \\ &\quad \uparrow \text{symmetry from} \\ &= \frac{-g^2}{2} \frac{\Gamma(\frac{d}{2}-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \left[ \frac{m^2 - p^2 x(1-x)}{\mu^2} \right]^{\frac{d}{2}-2} \cdot (\mu^{2-\frac{d}{2}+\epsilon}) \end{aligned}$$

$$-\text{O} = -\frac{1}{2} i g \mu^{\frac{d-2}{2}} \cdot \frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} (m^2)^{1-\frac{d}{2}}$$

$\therefore$  For  $d=4$

$$-\text{O} = \frac{1}{2} \frac{g^2}{(4\pi)^2} \left\{ \frac{1}{\epsilon} - r_\epsilon + \log 4\pi + \dots \right\}$$

$$-\text{O} = +\frac{1}{2} i \frac{g}{(6\pi)^2} m^2 \left\{ \frac{1}{\epsilon} - r_\epsilon + \log 4\pi + \dots \right\}$$

$$\therefore \delta_p = 0, \quad \delta_{m^2} = -\frac{1}{2} \frac{g^2}{(4\pi)^2} \frac{1}{\epsilon}, \quad (P_3)$$

$$\delta_{\text{tadpole}} = \frac{1}{2} \frac{g}{(4\pi)^2} \frac{1}{\epsilon} \frac{1}{m^2}. \quad (\text{See TN\#4 Solutions.})$$

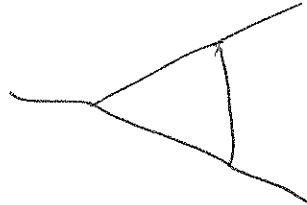
(d) In  $d=6$ ,  $D=6-2N$ .  $(10, 13)$  in P&S

$$N=1 \quad \text{---} \textcircled{M} \quad , \quad D=4$$

$$N=2 \quad \text{---} \textcircled{M} \text{---} \quad D=2.$$

$$N=3 \quad \text{---} \textcircled{M} \text{---} \quad D=0.$$

The new diagram is



$$\text{From (c)} = -i \frac{(eg)^3 \mu^{3\epsilon}}{(2\pi)^d} \int d^d k \cdot \frac{i}{k^2 - m^2 + i\epsilon} \frac{i}{(p+k)^2 - m^2 + i\epsilon} \frac{i}{(q+k)^2 - m^2 + i\epsilon}$$

$$\stackrel{d=6-2\epsilon}{=} (eg\mu^\epsilon) \cdot \left( \frac{1}{2} \frac{g^2}{(4\pi)^3} \right) \left\{ \frac{1}{\epsilon} - r_\epsilon + \dots \right\}$$

$$\therefore \delta_g = \frac{1}{2} \frac{g^2}{(4\pi)^3} \frac{1}{\epsilon} \cdot (eg\mu^\epsilon)$$

From (c) we get in  $d=6$

$$\delta_{p^2} = +\frac{1}{2} \frac{g^2}{(4\pi)^3} \frac{1}{6} \frac{1}{\epsilon}, \quad \delta_{\text{tadpole}} = \frac{1}{4} eg \frac{1}{m^4} \frac{1}{(4\pi)^3} \frac{1}{\epsilon}.$$

$$\delta_{m^2} = -\frac{1}{2} \frac{g^2}{(4\pi)^3} \frac{1}{\epsilon}.$$

(e). In  $d=8$ , we again use again the expression in (c). P.4

$$\begin{aligned}
 -O &= \frac{g^2}{d} \frac{P(-2+\varepsilon)}{(4\pi)^{4-\varepsilon}} \int_0^1 dx \left[ \frac{m^2 - p^2 x(-x)}{m^2} \right]^{2-\varepsilon} (u^2)^{d-\frac{d}{2}-\varepsilon} \\
 &= \frac{g^2}{2} \frac{1}{(4\pi)^4} \frac{1}{d} \int_0^1 dx \left[ \frac{m^2 - p^2 x(-x)}{m^2} \right]^2 \left[ \frac{1}{\varepsilon} - r_\varepsilon + \dots \right] \\
 &= -\frac{g^2}{4} \frac{1}{(4\pi)^4} \frac{m^4}{\varepsilon} \left\{ \frac{m^4}{m^4} - \frac{1}{6} \frac{\partial \cdot \partial m^2 p^2}{m^4} + \frac{p^4}{m^4} \frac{1}{5} \right\} \\
 &= -\frac{g^2}{4} \frac{1}{(4\pi)^4} \frac{1}{\varepsilon} \left\{ m^4 - \frac{1}{3} m^2 p^2 + \frac{1}{5} p^4 \right\}
 \end{aligned}$$

Note that  $[g] = -1$  in  $d=8$

$$\therefore \delta m = -\frac{g^2}{4} \frac{m^4}{(4\pi)^4} \frac{1}{\varepsilon}$$

$$\delta p^2 = +\frac{g^2}{2} \frac{m^2}{(4\pi)^4} \frac{1}{\varepsilon}.$$

We also need  $\delta p^4 = -\frac{g^2}{20} \frac{1}{(4\pi)^4} \frac{1}{\varepsilon}$ .

which requires a counter term of the

form  $(\partial \partial^2 \phi)^2$  !

- (f).  $\phi^3$  is superrenormalizable in  $d=4$  (relevant)  
 renormalizable in  $d=6$  (marginal)  
 non-renormalizable in  $d=8$  (irrelevant)

(P15)

(h) In  $d=4$ ,  $\bar{z}_g = 1 \therefore \beta = 0$   
 $\phi^3$  stays relevant.

### Problem 3

(a)

$$m_{\text{Dir}} = \frac{i(g_m g_{\mu\nu})}{(g_m g^2 - g_{\mu\nu} g^0)} \Pi(g^2)$$

$$\Pi(g^2) = -\frac{\alpha}{3\pi} \left\{ \frac{1}{\epsilon} - r_c + \log 4\pi + b \int_0^1 dx x(1-x) \log \left( \frac{M_4^2 - g^2 x(1-x)}{\mu^2} \right) \right\}$$

$$(b). \quad \text{Tr}(g^2) = \Pi(g^2) - \Pi(-M^2)$$

$$= -\frac{2\alpha}{\pi} \int_0^1 dx (1-x) x \log \left[ \frac{M_4^2 - g^2 x(1-x)}{M_4^2 + M^2 x(1-x)} \right]$$

$$\text{then } \text{Tr}(-M^2) = 0$$

$$(c). \quad \beta(e) = -\frac{e}{2} \frac{d}{de} \left[ e \text{Tr}(-M^2) \right] = \frac{2e \cdot \alpha}{\pi} \int_0^1 dx (1-x) x \frac{M^2 x(1-x)}{M_4^2 - M^2 x(1-x)}$$

$$\sim \frac{e^3}{12\pi^2} \quad M_4 \rightarrow 0$$

$$\sim \frac{e^3}{60\pi^2} \frac{M^2}{M_4^2}, \quad M_4 \rightarrow \infty$$

(P6)

(d). In MS,

$$\text{Tr}(g^2) = -\frac{\alpha}{3\pi} \left\{ -\epsilon + \log 4\pi + 6 \int_0^1 dx x(1-x) \log \left( \frac{M_F^2 g^2}{\mu^2} \delta(x) \right) \right\}$$

$$Z_3 = 1 + \delta_3 = 1 - \frac{\alpha}{3\pi} \frac{1}{\epsilon}.$$

$$\text{Ward identity} \Rightarrow Z_3 = \frac{1}{\sqrt{Z_g}} \Rightarrow Z_g = 1 + \frac{\alpha}{6\pi} \frac{1}{\epsilon}.$$

$\beta(\epsilon) = -\epsilon + \alpha \epsilon^2 \frac{d}{d\epsilon} Z_g^{(0)} = \frac{\epsilon^3}{12\pi^2}.$  → this is the same as  $\beta(\epsilon)$  for a massless fermion!

However, for  $M_F^2 \gg \mu^2$ , there's a large log in  $\text{Tr}(g^2)$ , and perturbation can not be trusted!

(e). From (7.74) in PQS, in full QED:

$$\langle 0 | T(A_\mu(x) A_\nu(y)) | 0 \rangle = \int \frac{d^4 k}{(2\pi)^4} e^{ik(x-y)} \frac{-i}{(1-\Pi(g^2))} \frac{g_{\mu\nu} - \frac{g_{\mu\nu} g^2}{g^2}}{k^2} + \frac{i}{g^2} \frac{\partial_{\mu}\partial_{\nu}}{k^2}.$$

In EFT,

$$\langle 0 | T(\partial_\mu(x) \partial_\nu(y)) | 0 \rangle = \int \frac{d^4 k}{(2\pi)^4} e^{ik(x-y)} \langle 0 | T(A_\mu(x) A_\nu(y)) | 0 \rangle$$

$$\stackrel{?}{=} \quad \partial_\mu \sim \frac{A_\mu}{\sqrt{1-\Pi(g^2)}} \rightarrow \frac{A_\mu}{\sqrt{1-\Pi(0)}} \\ \sim A_\mu \left( 1 + \frac{1}{2} \Pi(0) \right) \approx A_\mu \sqrt{1+\Pi(0)}$$

(P.7)

We Should choose  $\mu \sim M_4$  so that  
there's no large log in  $\text{Tr}(0)$ !

(+) From the form of the covariant derivative

$$\partial_\mu - i e g_F A^\mu \Rightarrow e_{\text{eff}} = \frac{e}{\sqrt{1 + \text{Tr}(0)}} \text{ if } A = A \cdot \sqrt{1 + \text{Tr}(0)} \\ = \partial_\mu - i e A$$

From the form of  $\text{Tr}(g)$ , if we choose

$$M_m^2 = M_4^2 \cdot e^{-\nu_E} \cdot 4\pi$$

then  $\text{Tr}(0) = 0$  at  $M_m^2$

$$\Rightarrow e_{\text{eff}}(M_m) = e(M_m).$$

$$(8) \quad \frac{d}{dm} e_{\text{eff}} = (\mu \frac{\partial}{\partial \mu} + \beta(e) \frac{\partial}{\partial e} + \gamma_{M_4} M_4 \frac{\partial}{\partial M_4}) e_{\text{eff}} = \beta_{\text{eff}}.$$

$$\beta(e) = \frac{2 \times e^3}{12\pi^2}$$

$$\gamma_M = -\frac{3e^2}{8\pi^2} \quad (\text{Computed in class. or see (12,112) in PSS})$$

$$\Rightarrow \cancel{\beta_{\text{eff}}} e_{\text{eff}} = e \left( 1 - \frac{1}{2} \text{Tr}(0) \right)$$

$$= e \left( 1 + \frac{d}{6\pi} \left( -\nu_E \log(4\pi t b) \int_0^1 dx x(1-x) \log \left( \frac{M_4^2}{\mu^2} \right) \right) \right)$$

(P.8)

$$\therefore \boxed{\beta_{\text{eff}}} M_0 \frac{\partial}{\partial e} \ell_{\text{eff}} = \frac{\alpha}{6\pi} \cdot 6 \cdot \frac{1}{6} (-\alpha) = -\frac{\alpha e}{3\pi} = -\frac{e^3}{12\pi^2}$$

$$\beta(e) \frac{\partial}{\partial e} \ell_{\text{eff}} = \beta(e) = 2 \times \frac{e^3}{12\pi^2}$$

$$r_{M_4} M_4 \frac{\partial}{\partial r_{M_4}} \ell_{\text{eff}} = -\frac{3e^2}{8\pi^2} \cdot \frac{\alpha}{6\pi} \cdot 6 \cdot \frac{1}{6} (+\alpha) \sim \mathcal{O}(e^4)$$

$$\therefore \beta_{\text{eff}} = -\frac{e^3}{12\pi^2} + 2 \frac{e^3}{12\pi^2} = \frac{e^3}{12\pi^2} \rightarrow \text{the tuning for one light fermi!}$$